Application of Ultradiscrete Kalman Filter to a Physical System

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Abstract

A new filtering method based on the procedure of ultradiscritization is applied to a nonlinear physical system. Numerical results again show the efficiency of the method.

1 Introduction

Kalman filter [1] is a powerful tool to estimate the state of noisy dynamical system from noisy measurement. It has been applied to various engineering problems. The filter is quite successful for linear systems. Although many attempts have been done to extend the filter to nonlinear systems (see, for example [2]), it seems that there is not any systematic way treating nonlinear systems.

In the preceding paper we have proposed a new filtering method for a nonlinear system [3], in which the original system is reduced to a piecewise linear one through the procedure of ultradiscretization [4], [5]. Then the discrete-time Kalman filter is readily applied to the obtained system by imposing some conditions on system variables and parameters. Although the nonlinear system we treated in the paper is rather artificial [6], the result of numerical experiment on the system shows the efficiency of the filter.

In this paper, we apply the new filter to a nonlinear physical system. It is an equation for nonlinear spring,

$$\frac{d^2x}{dt^2} + ax + bx^3 = 0, \tag{1}$$

where $x = x(t)$ is the displacement of a mass and a, b are system parameters. The parameter a is positive and b is positive for the hard spring and negative for the soft one, respectively. It is remarked that (1) has a conserved quantity

$$H(t) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4. \tag{2}$$

In section 2, we introduce a discrete analogue of (1) and rewrite the resulting equation in a simultaneous system. Then we consider signal and observing processes for the system variables by introducing a class of noises. Through the procedure of ultradiscretization with parity variables [5], these processes are reduced to piecewise linear forms, which we call ultradiscrete processes. In section 3, we first give a brief summary for the discrete time Kalman filter. Then we construct a Kalman filter for the ultradiscrete processes. We consider three different types of observing processes. Some results of numerical experiments for the filter are presented in section 4. Finally, concluding remarks are given in section 5.

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2 Nonlinear system and its ultradiscrete analogue

A discrete analogue of (1) given in [7] is written as

$$\frac{1}{\delta^2}\{x(t+\delta)-2x(t)+x(t-\delta)\}+2c_1\{x(t+\delta)+x(t-\delta)\}+4c_2x(t)+2c_3x(t)^2\{x(t+\delta)+x(t-\delta)\}=0 \tag{3}$$

or

$$x(t+\delta)=\frac{2(1-2c_2\delta^2)x(t)}{1+2c_1\delta^2+2c_3\delta^2x(t)^2}-x(t-\delta), \tag{4}$$

where $c_1+c_2=a/4$ and $c_3=b/4$. The difference eq.(3) has a conserved quantity

$$H(t)=\frac{1}{2\delta^2}\{x(t)-x(t-\delta)\}^2+c_1\{x(t)^2+x(t-\delta)^2\}+2c_2x(t)x(t-\delta)+c_3x(t)^2x(t-\delta)^2, \tag{5}$$

which corresponds to (2) and shows the integrability of (3).

Let us transform (4) into a simultaneous system,

$$\begin{cases}\tilde{x}^0_{k+1}=\tilde{x}^1_k\\ \tilde{x}^1_{k+1}=\dfrac{2(1-2c_2\delta^2)\tilde{x}^0_{k+1}}{1+2c_1\delta^2+2c_3\delta^2(\tilde{x}^0_{k+1})^2}-\tilde{x}^0_k\end{cases} \tag{6}$$

where $\tilde{x}^0_k=x(t)$ and $\tilde{x}^1_k=\tilde{x}^0_{k+1}=x(t+\delta)$. We now consider the signal process,

$$\begin{cases}\tilde{x}^0_{k+1}=\tilde{u}^0_k\tilde{x}^1_k\\ \tilde{x}^1_{k+1}=\tilde{u}^1_k\left\{\dfrac{2(1-2c_2\delta^2)\tilde{x}^0_{k+1}}{1+2c_1\delta^2+2c_3\delta^2(\tilde{x}^0_{k+1})^2}-\tilde{x}^0_k\right\},\end{cases} \tag{7}$$

where $\{\tilde{u}^0_k\}$ and $\{\tilde{u}^1_k\}$ are white noises with average 1 and sufficiently small variance.

For the observing process, we consider three different types:

$$\text{type 1}\quad \begin{cases}\tilde{y}^0_k=\tilde{w}^0_k\tilde{x}^0_k\\ \tilde{y}^1_k=\tilde{w}^1_k\tilde{x}^1_k\end{cases} \tag{8}$$

$$\text{type 2}\quad y^0_k=\tilde{w}^0_k\tilde{x}^0_k\tilde{x}^1_k \tag{9}$$

$$\text{type 3}\quad y^0_k=\tilde{w}^0_k(\tilde{x}^0_k+\tilde{x}^1_k), \tag{10}$$

where $\{\tilde{w}^0_k\}$ and $\{\tilde{w}^1_k\}$ are again white noises with average 1 and sufficiently small variance.

In order to ultradiscretize (7)-(10), we introduce varible transformations with $\varepsilon>0$,

$$\tilde{x}^i_k=\xi^i_ke^{X^i_k/\varepsilon},\quad \tilde{y}^i_k=\eta^i_ke^{Y^i_k/\varepsilon},\quad \tilde{u}^i_k=e^{U^i_k/\varepsilon},\quad \tilde{w}^i_k=e^{W^i_k/\varepsilon},\quad \delta=e^{\Delta/\varepsilon},\quad c_i=e^{\alpha_i/\varepsilon},$$

where ξ^i_k and η^i_k are sign variables. It is noted that noise variables are always positive due to the assumption on them. It is also noted that all of the parameters $c_i, i=1,2,3$ are assumed to be positive which correspond to the case of hard spring. For the simplification, we put

$$\breve{\alpha}_i:=2\Delta+\alpha_i,$$
$$A_k:=\max[0,\breve{\alpha}_1,2X^0_k+\breve{\alpha}_3].$$

Then by applying ultradiscretization with parity variables, we obtain ultradiscrete signal process,

$$\begin{cases} \xi_{k+1}^0 = \xi_k^1 \\ X_{k+1}^0 = X_k^1 + U_k^0 \end{cases} \tag{11}$$

$$\max \begin{bmatrix} (\xi_{k+1}^1) + X_{k+1}^1 + A_{k+1} - U_k^1 \\ S(-\xi_{k+1}^0) + X_{k+1}^0 \\ S(\xi_{k+1}^0) + X_{k+1}^0 + \check{\alpha}_2 \\ S(\xi_k^0) + X_k^0 + A_{k+1} \end{bmatrix} = \max \begin{bmatrix} S(-\xi_{k+1}^1) + X_{k+1}^1 + A_{k+1} - U_k^1 \\ S(\xi_{k+1}^0) + X_{k+1}^0 \\ S(-\xi_{k+1}^0) + X_{k+1}^0 + \check{\alpha}_2 \\ S(-\xi_k^0) + X_k^0 + A_{k+1} \end{bmatrix} \tag{12}$$

where the function $S : \{1, -1\} \to \{0, -\infty\}$ is defined by

$$S(\xi) = \begin{cases} 0 & (\xi = 1) \\ -\infty & (\xi = -1), \end{cases} \tag{13}$$

and ultradiscrete observing process,

$$\begin{cases} \eta_k^i = \xi_k^i \\ Y_k^i = X_k^i + W_k^i \end{cases} \qquad (i = 0, 1) \tag{14}$$

for type 1,

$$\begin{cases} \eta_k^0 = \xi_k^0 \xi_k^1 \\ Y_k^0 = X_k^0 + X_k^1 + W_k^0 \end{cases} \tag{15}$$

for type 2 and

$$\begin{cases} \eta_k^0 = \begin{cases} \xi_k^0 & (X_k^0 \geq X_k^1) \\ \xi_k^1 & (X_k^0 < X_k^1) \end{cases} \\ Y_k^0 = \max(X_k^0, X_k^1) + W_k^0 \end{cases} \tag{16}$$

for type 3, respectively. It is remarked that, in the observing process of type 3, observed value can be indefinite, when $\xi_k^0 \xi_k^1 = -1$ and $X_k^0 = X_k^1$. However, we consider that it never happens in the noisy system.

3 Kalman filter for ultradiscrete processes

Before applying the Kalman filter to ultradiscrete signal and observing processes, we give a brief summary of the discrete-time Kalman filter. Let us consider that an n-vector state $\boldsymbol{x}_k$ follows the linear dynamical system disturbed by an r-vector white noise $\boldsymbol{u}_k$,

$$\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k \quad (k = 0, 1, 2, \ldots), \tag{17}$$

where an $n \times n$ matrix A_k and an $n \times r$ matrix B_k are given nonrandom quantities. We assume that the mean vector $\bar{\boldsymbol{u}}_k$ and the covariance matrix U_k of the noise vector $\boldsymbol{u}_k$ are prescribed. We do not know $\boldsymbol{x}_k$ itself but obtain an m-vector observation $\boldsymbol{x}'_k$ through the observing process

$$\boldsymbol{x}'_k = C_k \boldsymbol{x}_k + \boldsymbol{w}_k \quad (k = 0, 1, 2, \ldots), \tag{18}$$

where C_k is a given $n \times m$ matrix and $\boldsymbol{w}_k$ is an m-vector white noise. We assume that the mean vector $\bar{\boldsymbol{w}}_k$ and the covariance matrix W_k of the noise vector $\boldsymbol{w}_k$ are given. We also assume that the mean $\bar{\boldsymbol{x}}_0$ and covariance X_0 of the initial value $\boldsymbol{x}_0$ are prescribed and that $\boldsymbol{x}_0$, $\{\boldsymbol{u}_k\}$ and $\{\boldsymbol{w}_k\}$ are independent.

Our problem is to obtain the best estimator $\hat{\boldsymbol{x}}_k$ from $\boldsymbol{x}'_0, \ldots, \boldsymbol{x}'_k$. The discrete-time Kalman filter gives an answer in the form of an explicit recurrence formula:

$$\hat{\boldsymbol{x}}_k = \tilde{\boldsymbol{x}}_k + P_k{}^t C_k W_k{}^{-1}(\boldsymbol{x}'_k - C_k \tilde{\boldsymbol{x}}_k - \bar{\boldsymbol{w}}_k) \quad (k = 0, 1, 2, \ldots), \tag{19}$$

where

$$\tilde{\boldsymbol{x}}_k = \begin{cases} \bar{\boldsymbol{x}}_0 & (k = 0) \\ A_{k-1}\hat{\boldsymbol{x}}_{k-1} + B_{k-1}\bar{\boldsymbol{u}}_{k-1} & (k = 1, 2, 3, \ldots) \end{cases} \tag{20}$$

$$P_k = \left(M_k{}^{-1} + {}^t C_k W_k{}^{-1} C_k\right)^{-1} \tag{21}$$

$$M_k = \begin{cases} \mathrm{X}_0 & (k = 0) \\ A_{k-1}P_{k-1}{}^t A_{k-1} + B_{k-1}U_{k-1}{}^t B_{k-1} & (k = 1, 2, 3, \ldots). \end{cases} \tag{22}$$

Note that P_k is the covariance matrix for the error $\hat{\boldsymbol{x}}_k - \boldsymbol{x}_k$ and that (21) is the discrete (matrix) Riccati equation.

It is worthwhile to comment on solutions of the ultradiscrete signal process (11) and (12) for the deterministic case without noises. If we give the parameters and initial values satisfying $\breve{\alpha}_2 < 0$, $\xi_0^0 \xi_0^1 = 1$, $X_0^0 < X_0^1$, $A_1 = 2X_1^0 + \breve{\alpha}_3$, $X_0^0 + X_0^1 + \breve{\alpha}_3 = 0$, then we have the four periodic solution,

$$\begin{pmatrix} (\xi_k^0, X_k^0) \\ (\xi_k^1, X_k^1) \end{pmatrix} = \begin{cases} \begin{pmatrix} (\xi_0^0, X_0^0) \\ (\xi_0^1, X_0^1) \end{pmatrix} & (k \equiv 0) \\ \begin{pmatrix} (\xi_0^1, X_0^1) \\ (-\xi_0^0, X_0^0) \end{pmatrix} & (k \equiv 1) \\ \begin{pmatrix} (-\xi_0^0, X_0^0) \\ (-\xi_0^1, X_0^1) \end{pmatrix} & (k \equiv 2) \\ \begin{pmatrix} (-\xi_0^1, X_0^1) \\ (\xi_0^0, X_0^0) \end{pmatrix} & (k \equiv 3). \end{cases} \tag{23}$$

In this case the ultradiscrete signal process without noises are expressed by

$$\begin{pmatrix} \xi_{k+1}^0 \\ \xi_{k+1}^1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \xi_k^0 \\ \xi_k^1 \end{pmatrix} \tag{24}$$

$$\begin{pmatrix} X_{k+1}^0 \\ X_{k+1}^1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_k^0 \\ X_k^1 \end{pmatrix}. \tag{25}$$

If we impose the same conditions on the parameters and initial values, the noisy signal process is expressed as

$$\begin{cases} \begin{pmatrix} \xi^0_{k+1} \\ \xi^1_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \xi^0_k \\ \xi^1_k \end{pmatrix} \\ \begin{pmatrix} X^0_{k+1} \\ X^1_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X^0_k \\ X^1_k \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U^0_k \\ U^1_k \end{pmatrix} \end{cases} \tag{26}$$

by assuming the noises do not affect on the max operatin. The noisy observing processes are also writtten as

$$\begin{cases} \begin{pmatrix} \eta^0_k \\ \eta^1_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^0_k \\ \xi^1_k \end{pmatrix} \\ \begin{pmatrix} Y^0_k \\ Y^1_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^0_k \\ X^1_k \end{pmatrix} + \begin{pmatrix} W^0_k \\ W^1_k \end{pmatrix} \end{cases} \tag{27}$$

for type 1,

$$\begin{cases} \eta^0_k = \xi^0_k \xi^1_k \\ Y^0_k = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} X^0_k \\ X^1_k \end{pmatrix} + W^0_k \end{cases} \tag{28}$$

for type 2 and

$$\begin{cases} \eta^0_k = \begin{cases} \xi^0_k & (X^0_k \geq X^1_k) \\ \xi^1_k & (X^0_k < X^1_k) \end{cases} \\ Y^0_k = \begin{pmatrix} \frac{1-(-1)^k}{2} & \frac{1+(-1)^k}{2} \end{pmatrix} \begin{pmatrix} X^0_k \\ X^1_k \end{pmatrix} + W^0_k \end{cases} \tag{29}$$

for type 3, respectively.

Let us construct the Kalman filter for these three types:

type 1

For the observing process (27), the filter is given by

$$\hat{\boldsymbol{X}}_k = \tilde{\boldsymbol{X}}_k + P_k {W_k}^{-1}(\boldsymbol{Y}_k - \tilde{\boldsymbol{X}}_k) \tag{30}$$

where

$$\tilde{\boldsymbol{X}}_k = A\hat{\boldsymbol{X}}_{k-1} \tag{31}$$

$$P_k = \left({M_k}^{-1} + {W_k}^{-1}\right)^{-1} \tag{32}$$

$$M_k = AP_{k-1}A + U_{k-1} \tag{33}$$

and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For the estimators of sign variables, it is adequate to take $\hat{\xi}^0_k = \eta^0_k$ and $\hat{\xi}^1_k = \eta^1_k$.

type 2

For the observing process (28), the filter is

$$\hat{\boldsymbol{X}}_k = \tilde{\boldsymbol{X}}_k + P_k{}^tCW_k{}^{-1}(Y_k^0 - C\tilde{\boldsymbol{X}}_k), \tag{34}$$

where

$$\tilde{\boldsymbol{X}}_k = A\hat{\boldsymbol{X}}_{k-1} \tag{35}$$

$$P_k = \left(M_k{}^{-1} + {}^tCW_k{}^{-1}C\right)^{-1} \tag{36}$$

$$M_k = AP_{k-1}A + U_{k-1}, \tag{37}$$

and where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$. For the estimators of sign variables, it is adequate to take $\hat{\xi}_k^0\hat{\xi}_k^1 = \eta_k^0$. Although $\hat{\xi}_k^0$ and $\hat{\xi}_k^1$ themselves are not determined, we can estimate them from the solutions of the deterministic case.

type 3

For the observing process (29), the filter is

$$\hat{\boldsymbol{X}}_k = \tilde{\boldsymbol{X}}_k + P_k{}^tC_kW_k{}^{-1}(Y_k^0 - C_k\tilde{\boldsymbol{X}}_k) \tag{38}$$

where

$$\tilde{\boldsymbol{X}}_k = A\hat{\boldsymbol{X}}_{k-1} \tag{39}$$

$$P_k = \left(M_k{}^{-1} + {}^tC_kW_k{}^{-1}C_k\right)^{-1} \tag{40}$$

$$M_k = AP_{k-1}A + U_{k-1} \tag{41}$$

and where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C_k = \begin{pmatrix} \frac{1-(-1)^k}{2} & \frac{1+(-1)^k}{2} \end{pmatrix}$. For the estimators of sign variables, it is adequate to take η_k^0 as $\hat{\xi}_k^i$ if $\max(\hat{X}_k^0, \hat{X}_k^1) = \hat{X}_k^i$, $i = 0$ or 1. Although the other sign variable is not determined, we can again estimate it from the solutions of the deterministic case.

4 Numerical results

In this section, we give some numerical results. We compute the original state variables through (26) and the estimators through (27) for type1, (28) for type2, and (29) for type3, respectively.

For simplicity, we generate all noises from $\mathcal{N}(0, 1/100)$, where $\mathcal{N}(\mu, \sigma)$ is the normal distribution with mean μ and standard deviation σ. We take the system parameters as

$$\breve{\alpha}_1 = -1,\ \breve{\alpha}_2 = -1,\ \breve{\alpha}_3 = 1. \tag{42}$$

For the initial values, we take

$$\xi_0^0 = +1,\ X_0^0 = 1, \quad \xi_0^1 = +1,\ X_0^1 = 2. \tag{43}$$

It is remarked that the initial value $\tilde{\boldsymbol{X}}_0$ should be the average of the initial signals, but we take the initial signals themselves and that M_0 should be the covariance marix of the initial signals, but instead we take the variance of noises $(1/100^2)$ times the unit matrix.

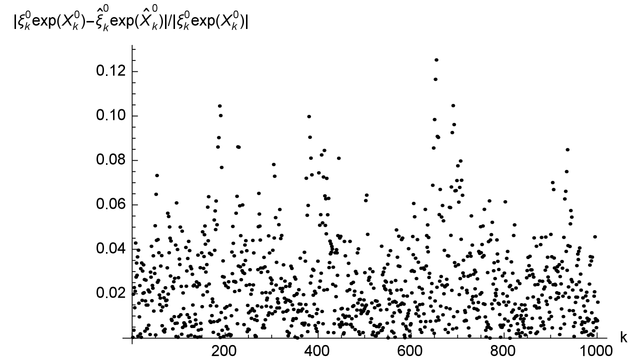

Figure 1: type1: time series of the relative error for $\xi_k e^{X_k^0}$

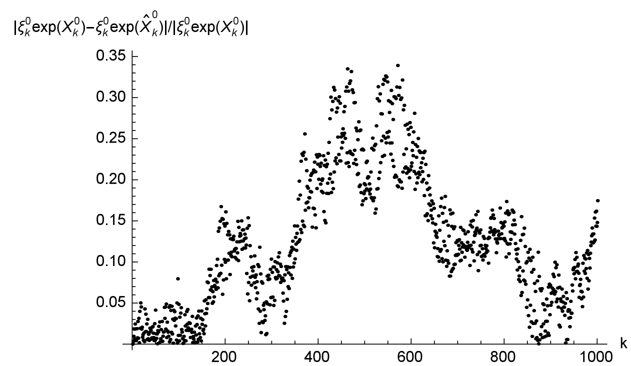

Figure 2: type2: time series of the relative error for $\xi_k e^{X_k^0}$

Figure 1 gives the time series of the relative error between the original signal $\xi_k e^{X_k^0}$ and the estimator $\hat{\xi}_k^0 e^{\hat{X}_k^0}$, $|\xi_k e^{X_k^0} - \hat{\xi}_k^0 e^{\hat{X}_k^0}|/|\xi_k^0 e^{X_k^0}|$, for the estimator of type1. The result shows that the error is almost always less than 10%.

Figures 2 and 3 are the numerical results for the estimator of type2. It is noted that the signs of the signals themselves are taken for those of the estimators. The results shows that the relative error for $\xi_k^0\xi_k^1 \exp(X_k^0 + X_k^1)$ is almost always less than 10%, although the relative error for $\xi_k e^{X_k^0}$ is rather large.

Figures 4 and 5 are the numerical results for the estimator of type3. It is noted that the signs of the signals are taken for those of the estimators. The results shows that the relative error for $\exp(\max(X_k^0, X_k^1))$ is again less than 10%, although the relative error for $\xi_k e^{X_k^0}$ is rather large.

5 Concluding remarks

We have applied the new filtering method based on the procedure of ultradiscritization to a physical system expressing nonlinear spring. For the system, we proposed an ultradiscrete signal process and three types of ultradiscrete observing processes. We would emphasize that the Kalman filter for the signal and observing processes can be constructed only by imposing some conditions on system variables and parameters.

Numerical result for the estimator of type1 shows fairly good efficiency of the filtering.

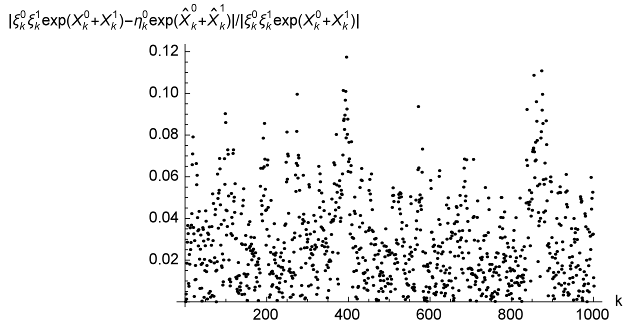

Figure 3: type2: time series of the relative error for $\xi_k^0\xi_k^1\exp(X_k^0+X_k^1)$

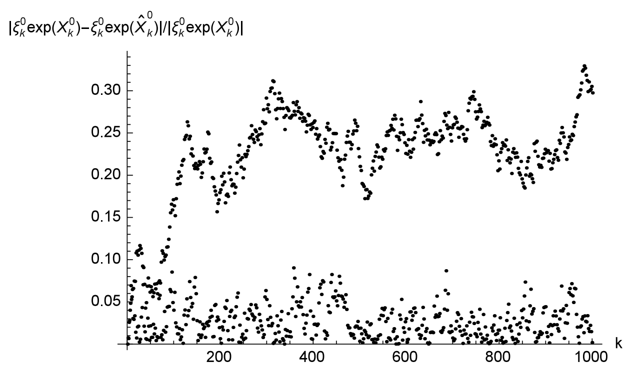

Figure 4: type3: time series of the relative error for $\xi_k^0\exp(X_k^0)$

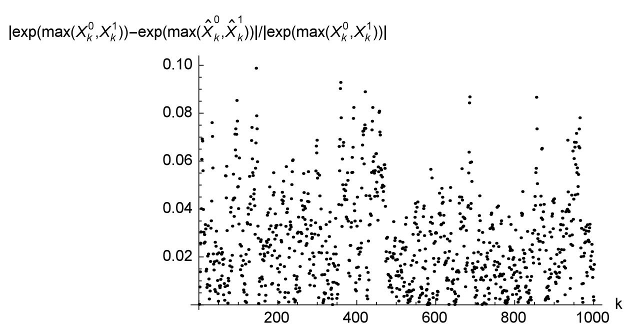

Figure 5: type3: time series of the relative error for $\exp(\max(X_k^0,X_k^1))$

The results for the other two cases are not as good as the type1, although we get rather good efficiency for some particular system variables. In order to get better filtering results, it may be important to consider another way of the discretization and suitable choice of the system variables. In a forthcoming paper, we will propose an improved ultradiscrete signal and observing processes and present better filtering results.

Since the procedure of constructing the Kalman filter is quite systematic, we may apply our method to more general problems. It is a future subject to consider much more practical situation.

acknowledgements
This research was supported by JSPS KAKENHI 24560078 and 26790082.

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(原稿提出: 2016 年 1 月 18 日; 修正稿提出: 2016 年 1 月 26 日)